Solids of Revolution with Minimum Surface Area, Part II

Skip Thompson
Department of Mathematics & Statistics
Radford University
Radford, VA 24142
thompson@radford.edu

Abstract

We consider the problem of determining the minimum surface area of solids obtained when the graph of a function or more general parametric curve is revolved about oblique lines. We develop a solution procedure that utilizes results from a previous study that considered the same problem for horizontal lines. We describe a refinement of the solution procedure that can be used to simplify the solution. We discuss several issues that must be considered for the question of revolving about oblique lines. We provide several plots and figures to illustrate these issues and to whet the appetite of students who my wish to explore them.

1 Introduction

In a previous study [8] we considered the question of determining the minimum surface area of solids obtained when the graph of a continuously differentiable function is revolved about horizontal lines. We found that for a given function $f(x)$ over the interval $a \leq x \leq b$, there is a unique horizontal line $y = k$ that yields the minimum surface area where surface area is defined to be

$$S(k) = 2\pi \int_a^b |f(x) - k| \sqrt{1 + (f'(x))^2}\, dx. \tag{1}$$

In this paper we consider the more general question in which a function of interest is revolved around oblique lines $y = mx + \beta$ (which we will refer to as axes of revolution and as minimal lines throughout this paper). We are interested in unsigned surface areas defined by

$$S(m, \beta) = \frac{2\pi}{\sqrt{m^2 + 1}} \int_a^b |f(x) - (mx + \beta)| \sqrt{1 + (f'(x))^2}\, dx. \tag{2}$$

We allow (in fact, require) the axis of revolution to intersect the graph of the function or parametric curve; and we wish to find which such line yields the minimum unsigned surface area.

2 Solution Procedure

It is straightforward to show that for the points at which the gradient of $S(m, \beta)$ vanishes, the Hessian also vanishes; so the usual test for minima is inconclusive. (Roughly speaking, the candidates for minima tend to lie along ridges making it difficult to work with Eq. 2 directly.) Therefore, rather than work directly with $S(m, \beta)$ and in order to make use of the results in [8], we proceed as follows. Following a clever lead in [9], we translate the origin to the point $(0, \beta)$ and then rotate the axes through an angle of $\theta = \tan^{-1}(m)$. This is accomplished using the transformation

$$\begin{aligned} w(x, y) &= \cos(\theta)\, x + \sin(\theta)\ (y - \beta) \\ z(x, y) &= -\sin(\theta)\, x + \cos(\theta)\ (y - \beta)\,. \end{aligned} \tag{3}$$

In the wz-coordinate system, the w-axis is the line $y = mx+\beta$, lines parallel to this line are horizontal, and the graph of $f(x)$ is given by the equations

$$\begin{aligned} w &= \cos(\theta)\, x + \sin(\theta)\ (f(x) - \beta) \\ z &= -\sin(\theta)\, x + \cos(\theta)\ (f(x) - \beta)\,. \end{aligned} \tag{4}$$

$w(x)$ and $z(x)$ represent a set of parametric equations for the function $f(x)$. Given a value of m, we can start with any value of β, say, $\beta = 0$. Our first task is to determine the value β_S which yields the minimum surface area S_m when the graph of $f(x)$ is revolved about the line $y = mx + \beta_S$. Our second task will be to then determine the value of m that minimizes S_m.

The proof given in [8] shows that for a given slope m, the surface area is concave up and has a unique minimum. It applies directly here since we're interested in rotating about horizontal lines in the wz-coordinate system. (It is also straightforward to prove this result directly in a manner similar to [8].) For a given value of m, the Golden Search procedure used in [8] can be used to find β_S and S_m. For this purpose the surface area is given by

$$2\pi \int_a^b |z(x) - k| \sqrt{(w'(x))^2 + (z'(x))^2}\, dx = 2\pi \int_a^b |z(x) - k| \sqrt{1 + (f'(x))^2}\, dx \tag{5}$$

for any horizontal line $z = k$. When it is rotated through an angle of $-\theta$, the horizontal line $z = k$ yielding the minimum surface area determines the line $y = mx + \beta_S$ where

$$\beta_S = \beta + k\sqrt{m^2+1}. \tag{6}$$

To see this set $z = k$ equal to $-\sin(\theta)x + \cos(\theta)(y - \beta)$ and divide by $\cos(\theta)$ to obtain

$$k \sec(\theta) = -\tan(\theta)x + y - \beta,$$

or equivalently,

$$k\sqrt{m^2+1} = -mx + y - \beta,$$

so that

$$y = mx + \beta + k\sqrt{m^2+1}.$$

Therefore, $\beta_S = \beta + k\sqrt{m^2+1}$ as claimed.

The next step is to minimize S_m. For each of the examples in Section 3, we have included plots of S_m. In each case S_m is not unimodal but has a clearly defined minimum that can be located using an outer Golden Search or other optimization procedure on an interval containing the minimum and on which S_m is unimodal. The results reported in this paper were obtained using nested Golden Searches for both minimizations.

2.1 Computational Difficulties

Though intuitively pleasing, the question of revolving about oblique lines is more difficult than revolving about horizontal lines. Several computational (and conceptual) issues have to be considered. This must be kept in mind when the figures and numerical results we give are interpreted.

If $f(x)$ is monotone, it and its reflection about a line $y = mx + \beta$ often are functions with respect to the line of revolution. However, this need not be the case. The reflection of $f(x)$ need not be a function with respect to the axis of revolution; and the parametric curve $C = (w(x), z(x))$ need not be a function in the wz-coordinate system. In this case the tangent line to the curve C will be vertical at one or more points. This occurs when $w'(x) = 0$ or equivalently, $f'(x) = -\dfrac{1}{m}$ so that the tangent line is perpendicular to the axis of revolution.

Note that Eq. 5 remains valid even if $z(x)$ is not a function in the wz-coordinate system. This is the case since we can parameterize the solids of revolution using

$$(w(x), |z(x) - k|\cos(\alpha), |z(x) - k|\sin(\alpha)),\ a \le x \le b,\ 0 \le \alpha \le 2\pi.$$

With this parameterization, the usual three-dimensional surface area formula reduces to Eq. 5.

We are trying to find a line that intersects the graph of $f(x)$ in order to minimize the surface area. We are thus allowing singular points [3] at which the line and function intersect. In this case when the revolution is performed, portions of the solid originating by revolving the parts of the curve below the line can "collide" with portions originating above the line, forcing us to admit self-intersecting surfaces [3].

Although the minimum surface of revolution for a well-behaved function need not be a self-intersecting surface, it is worth pointing out that allowing the axes of revolution to intersect the graph of $f(x)$ does in fact *always* introduce such surfaces. To see this assume that $f(x)$ is any continuously differentiable function on $[a, b]$. Pick any value of x_0 for which $f'(x) \neq 0$ and $f''(x) \neq 0$ and let $m = -\dfrac{1}{f'(x)}$. Now consider the line L defined by $y = mx + \beta$ containing the point $(x_0, f(x_0))$ and perpendicular to the tangent line TL at this point. After the rotation of axes through $\theta = \tan^{-1}(m)$ is performed, L is the w-axis and the TL is the z-axis. The curve C has the vertical tangent line TL at $(x_0, f(x_0))$. Since $(x_0, f(x_0))$ is not an inflection point, surface collisions corresponding to portions of C above and below L must occur near $(x_0, f(x_0))$, leading to self-intersecting surfaces, Again our task is to find the line $y = mx + \beta_S$ parallel to L that yields the minimum surface of revolution.

Another problem can arise since portions of the surface area may be duplicated by Eq. 2. This problem can also occur when portions of $f(x)$ are symmetric with respect to the axis of revolution. These issues become even more important when we deal with general parametric curves rather than functions since we must also deal with closed curves and loops.

It is instructive to see the manner in which our solids of revolution differ from the usual ones when the axis of rotation intersects the graph of $f(x)$ and what unsigned surface area is obtained. We use a simple example to illustrate this. Suppose we are interested in revolving the graph of $f(x) = x^2$ for the interval $[-1, 1]$ about the vertical line $x = -\dfrac{1}{2}$. (a) in Fig. 1 depicts the axis of revolution (black), $f(x)$ (blue), and the reflection of $f(x)$ about the axis of revolution (red). (b)-(d) depict the solids of revolutions obtained by separately revolving the portions of the graph of $f(x)$ (b) for the interval $[0, 1]$, (c) for the interval $[-1, 0]$, and (d) for the interval $[-1, 1]$ about this axis. Solid (c) intersects the axis of revolution at the point $(-1/2, 1/4)$ and is contained inside solid (b). The bottom portion of solid (c) results from revolving $f(x)$ for the interval $[-1/2, 0]$; and the top portion results from revolving $f(x)$ for the interval $[-1, -1/2]$. Solid (b) has a surface area of 9.975 and solid (c) has a surface area of 2.375. Solid (d) is the one of interest; its surface area 12.352 is the sum of these surface areas.

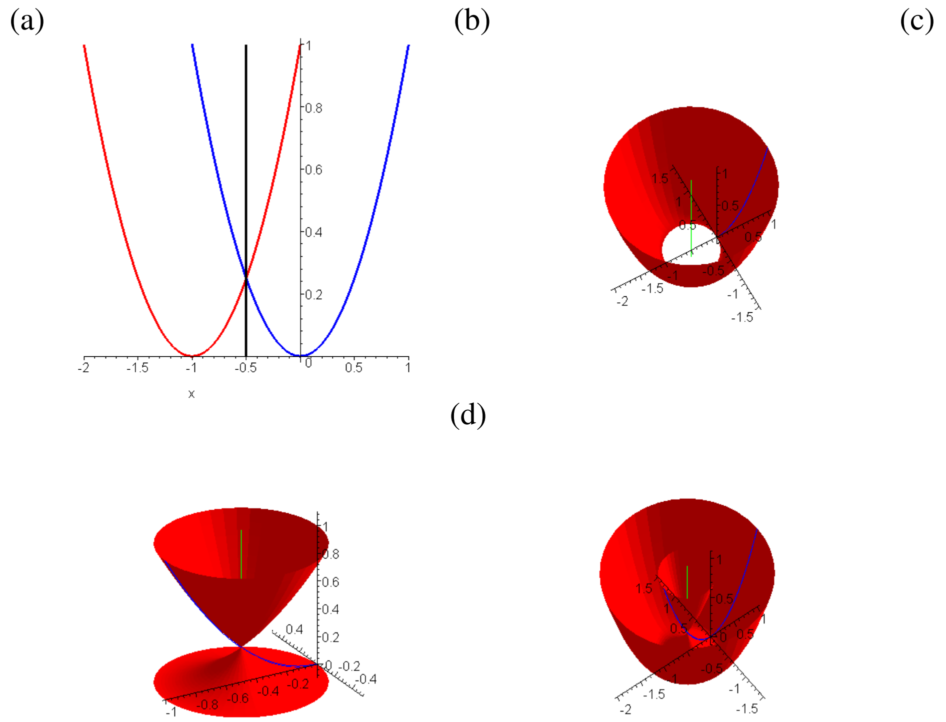


Figure 1: $f(x) = x^2, -1 \leq x \leq 1$

To illustrate the question of duplicated area, consider the function $f(x)$ where $f(x) = x^4$ for $0 \leq x \leq 1$ and $f(x) = \alpha x^4$ for $-1 \leq x < 0$. Denote by S_α the surface area obtained when the graph of this function is revolved about the y-axis. If $\alpha = 1$, Eq. 2 will yield a value of S_1 which is twice the actual surface area. Note that if α is slightly less than 1, S_α is very close to S_1 and does represent the area of the surface of revolution in question. In fact, we can think of the doubled surface area S_1 as the limiting surface area as $\alpha \to 1$.

The question of what computing platform should be used deserves mention. Although a more efficient implementation of our solution procedure in a high level language such as Fortran 90/95 could easily be developed, it should become apparent to the reader that the availability of a CAS such as Maple or Mathematica is invaluable for studying the questions of interest.

2.2 Simplifying the Solution Procedure

Because a nested Golden Search optimization is employed, Maple execution times can be quite long due to the number of integrations required. Fortunately, the time can be reduced significantly (in many cases from hours to minutes!) in the following manner. In the event one of the axes of revolution intersects the graph of $f(x)$ exactly once, the surface area is given by

$$2\pi \int_a^c (z(x)-k)\sqrt{(w'(x))^2+(z'(x))^2}\,dx \;-\; 2\pi \int_c^b (z(x)-k)\sqrt{1+(f'(x))^2}\,dx \tag{7}$$

or its negative (depending on whether $(a, f(a))$ is above or below the line) where c is the value of x for which the line intersects the graph of $f(x)$. Differentiation yields

$$\frac{\partial S}{\partial k} = -\frac{2\pi}{\sqrt{m^2+1}}\left(\int_a^c \sqrt{1+(f'(x))^2}\,dx - \int_c^b \sqrt{1+(f'(x))^2}\,dx\right),$$

Setting this expression to 0 and solving for c shows that c is the value for which the arc length of $f(x)$ from a to c is equal to the arc length from c to b. For a given function, c may be calculated using a nonlinear equation solver. The minimal axes of revolution that intersect $f(x)$ once pivot around the point $(c, f(c))$ as m is varied. For a given value of m, the corresponding value of β_S may be approximated using this pivot point and m, thereby eliminating the need to perform the inner Golden Search to determine β_S. In some cases this actually yields the correct minimum surface area (even when the line intersects the graph of $f(x)$ more than once). In other cases it determines an area that is close to the actual minimum for the problem as we will see in Section 3.

3 Examples

For each of the examples in this section, the following are included.

1. An animated plot showing the minimal surfaces as m is varied in equal increments from -5 to 5
2. An animated plot showing the minimal axes of revolution $y = mx + \beta_S$ for the same values of m
3. A plot of the solid with minimum surface area obtained using the nested search procedure
4. A plot of S_m

All results given were obtained using Maple [5]. The surfaces of revolution were plotted in parametric form using

```
plot3d([wofx(x),(zofx(x)-kminS)*cos(t),
        (zofx(x)-kminS)*sin(t)],x=a..b,
        t=0..2*Pi,axes=framed,labels=["w","z",""],
        scaling=CONSTRAINED,glossiness=1,
        numpoints=ngridw*ngridz,
        lightmodel=light2):
```

where wofx(x) and zofx(x) are the functions $w(x)$ and $z(x)$, ngridw = 51 and ngridz = 51 are the numbers of grid points, and kminS is the horizontal yielding the minimum surface area in the wz-coordinate system.. Integrations were performed using a Maple adaptation Adaptmw.mws of the Adapt.m integration algorithm from [6]. The solutions of the nonlinear equations to determine the pivot values discussed in Section 2.2 were performed using a Maple adaptation Zeromw.mws of the Zero.m nonlinear equation solver from [6].

3.1 Example 1: $f(x) = 2\sin(x) - \cos(2x),\ 0 \leq x \leq 2\pi$

For this function a minimum surface area of $S = 62.6$ is obtained for $m = -0.71$ and $\beta_S = 2.33$. The simplified solution procedure yields an approximate minimum of $S = 68.4$ for $m = -0.98$ and $\beta_S = 3.63$. The minimal lines $y = mx + \beta_S$ that intersect the graph of $f(x)$ once pivot around the point $(c, f(c))$ where $c = 2.388$. Other minimal lines intersect $f(x)$ one, two, or three times and don't pivot around this point.

(a) (b)

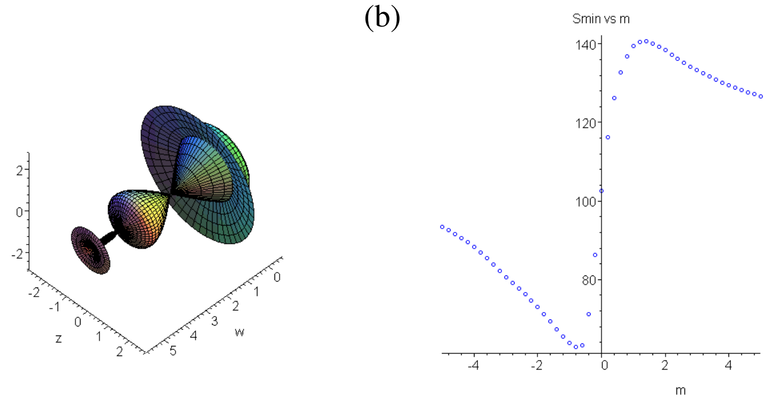


Figure 2: (a) Minimum Surface of Revolution and (b) S_m for Example 1

Figure 3: Surfaces of Revolution for Example 1

Figure 4: Axes of Revolution for Example 1

3.2 Example 2: $f(x) = x\sin(x),\ 0 \leq x \leq 2\pi$

For this function a minimum surface area of $S = 82.8$ is obtained for $m = -1.73$ and $\beta_S = 4.99$. The simplified solution procedure yields an approximate minimum of $S = 88.6$ for $m = -2.17$ and $\beta_S = 5.84$. The minimal lines $y = mx + \beta_S$ that intersect the graph of $f(x)$ once pivot around the point $(c, f(c))$ where $c = 3.861$. Other minimal lines intersect $f(x)$ three times and don't pivot around this point.

(a) (b)

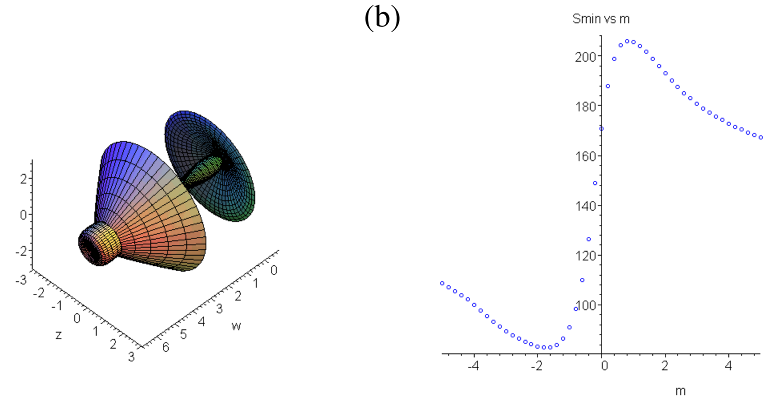


Figure 5: (a) Minimum Surface of Revolution and (b) S_m for Example 2

Figure 6: Surfaces of Revolution for Example 2

Figure 7: Axes of Revolution for Example 2

3.3 Example 3: $f(x) = x\cos(x),\ 0 \leq x \leq 3\pi$

For this function a minimum surface area of $S = 369$ is obtained for $m = -3.92$ and $\beta_S = 27.3$. The minimal lines $y = mx + \beta_S$ that intersect the graph of $f(x)$ once pivot around the point $(c, f(c))$ where $c = 6.456$. Other minimal lines intersect $f(x)$ more than once and don't pivot around this point.

(a) (b)

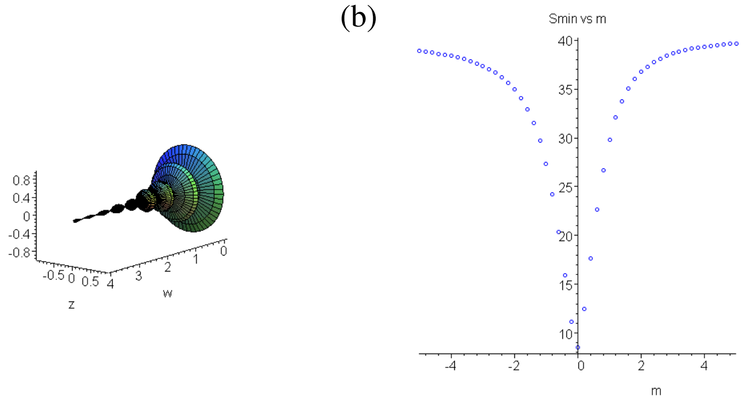


Figure 8: (a) Minimum Surface of Revolution and (b) S_m for Example 3

Figure 9: Surfaces of Revolution for Example 3

3.4 Vertical Rotations

If the function in Example 3.3 is changed to $f(x) = x\cos(2x),\ 0 \leq x \leq 3\pi$, a different behavior is observed. Unlike the previous examples, the minimum surface area is not obtained for a value of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Rather, as θ approaches $\pm\frac{\pi}{2}$, the surface area approaches the minimum surface area obtained by revolving $f(x)$ about vertical lines. This can be seen by using very large values of m or very small values of $-m$ in the minimization procedure described above. Alternatively, the precise definitions for w and z for $\theta = \pm\frac{\pi}{2}$ can be used to determine the vertical line $x = k$ that minimizes the surface area. For example, if $\theta = \frac{\pi}{2}$, $w(x)$ and $z(x)$ are given by $w = f(x) - \beta$ and $z = -x$, respectively. The oblique revolution calculations can be modified to accommodate these definitions directly. (Both approaches are used in Item 2 of Section 6.) The minimum surface area of 718 for

Figure 10: Axes of Revolution for Example 3

this problem is obtained by revolving $f(x)$ about the vertical line $k = -6.73$ (that is, $x = 6.73$). It is worth noting that for $\theta = \pm\frac{\pi}{2}$, the surface area integral reduces to

$$S = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2}\, dx$$

which is equivalent to the usual formula for calculating the surface area obtained by revolving $f(x)$ about the y-axis. Item 3 of Section 6 includes several other examples for which vertical revolution yield the minimum surface area.

4 General Parametric Curves

With appropriate changes, the solution procedure can be modified to allow minimization of the surface area obtained by revolving more general parametric curves $C = (x(t), y(t))$ about oblique lines. We define the surface area of interest in this case to be

$$2\pi \int_a^b |z(t) - k| \sqrt{(w'(t))^2 + (z'(t))^2}\, dt \tag{8}$$

where

$$\begin{aligned} w(t) &= \cos(\theta)\, x(t) + \sin(\theta)\ (y(t) - \beta) \\ z(t) &= -\sin(\theta)\, x(t) + \cos(\theta)\ (y(t) - \beta)\,. \end{aligned} \tag{9}$$

We illustrate this using the parametric curve in Example 10 of [9]. The curve is defined by

$$(x(t), y(t)) = (\sin(3t)\ \cos(t), \sin(3t)\ \sin(t)),\ 0 \le t \le \frac{\pi}{2}.$$

Fig. 11 shows the curve $(x(t), y(t))$, the original line of rotation $y = -0.9x - 1.378$, and tangent lines to the curve from the line. It also shows the line that yields the minimum surface of rotation for this

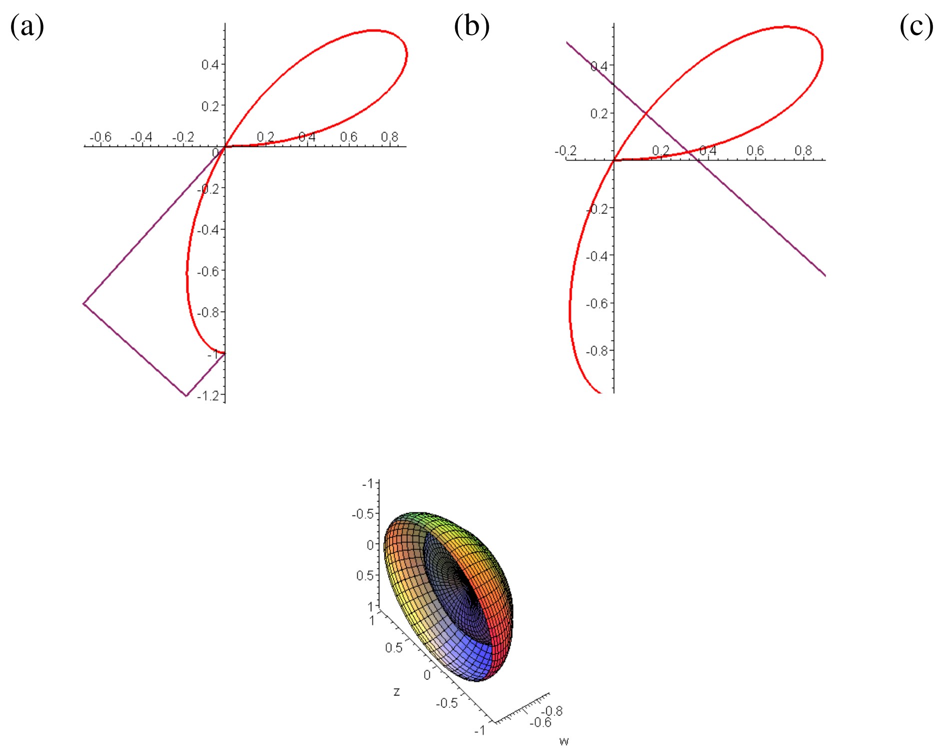


Figure 11: $(x(t), y(t)) = (\sin(3t)\ \cos(t), \sin(3t)\ \sin(t))\ 0 \leq t \leq \dfrac{\pi}{2}$.

value of m and the minimum surface of rotation. Revolving about the original line yields a surface area of 25.2 while revolving about the latter line yields a minimum surface area of 9.5.

By varying the right endpoint b we see that the minimal axes of revolution yielding the minimum surface areas move around to reflect the shape of the parametric curve. For example, if $b = \dfrac{\pi}{2}$ so that the curve contains one and a half leaves as in [9], we obtain $m = 1.256$, $\beta_S = -0.280$, and $S_{\mathtt{min}} = 3.19$. If b is changed to $\dfrac{\pi}{3}$ so that the parametric curve only contains one leaf, we obtain $m = 0.590$, $\beta_S = -0.471$, and $S_{\mathtt{min}} = 1.74$. If b is changed to $\dfrac{2\pi}{3}$ so that the parametric curve contains two leaves, we obtain $m = 1.738$, $\beta_S = -0.490$, and $S_{\mathtt{min}} = 4.80$. If b is changed to π so that the parametric curve contains three leaves, we obtain $m = 0$, $\beta_S = 0.075$, and $S_{\mathtt{min}} = 14.274$. Finally, if b is changed to $b = 0.4679647278306630$ so that the curve contains only one half of a leaf, we obtain $m = 0.423$, $\beta_S = -0.087$, and $S_{\mathtt{min}} = 0.23$. Fig. 12 shows the surfaces of revolution as the slope is varied from -5 to 5 for $b = \pi$. Item 3 of Section 6 is a Maple worksheet to minimize the surface area for this and several other parametric curves.

Recall our earlier remarks about self-intersecting surfaces. They are relevant here because the surface of revolution plots can be misleading. To see this, consider the curve defined by

$$(x(t), y(t)) = (\sin(2t), 2\sin(t)),\ \ 0 \leq t \leq 2\pi.$$

Figure 12: $(x(t), y(t)) = (\sin(3t)\cos(t), \sin(3t)\sin(t)),\ 0 \leq t \leq \pi$

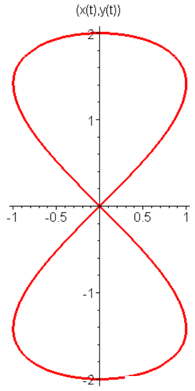
Fig. 13 displays the curve, the surface of revolution for $m = 5$ and $\beta = 0$, and half the surface of revolution for $b = 2\pi$. While it is not apparent in the former plot, the latter plot clearly shows the formation of the inner portion of the surface. Also recall our remarks regarding the fact that Eq. 2 duplicates surface area in the event any portion of the curve is symmetric with respect to the axis of revolution for $b = 2\pi$. For this curve, plotting either the surface of revolution about the x-axis or plotting only half of it for $m = 0$ and $b = 2\pi$ yields the same plot. Of course, this is due to fact that the curve is symmetric about the x-axis. If the curve is restricted so that $0 \leq t \leq \pi$, plotting half of the surface yields what we expect. Fig. 14 depicts the resulting surface and the expected half surface.

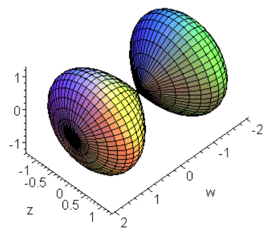
The manner in which nested surfaces develop can be quite interesting. Consider the following "barbed wire" curve defined by

$$(x(t), y(t)) \;=\; (t + \cos(2t),\, t - \sin(4t)),\ -5 \leq t \leq 5.$$

This curve is depicted in Fig. 15. Figs. 17 and 16 depict the minimal axes of revolution and minimum surfaces of revolution for the values $m = -1$, $m = 0$, and $m = 1$. (An overall minimum surface area of 82.7 is obtained for the values $m = 1.055$ and $\beta_S = 0.822$.) We find another type of animation useful for understanding the development of the solids, Figs. 18, 19, and 20 show the development of the three solids in increments of $\theta = \dfrac{\pi}{6}$. The figures clearly show the development of the respective solids.

(a) (b) (c)





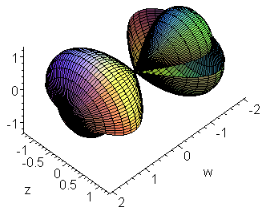
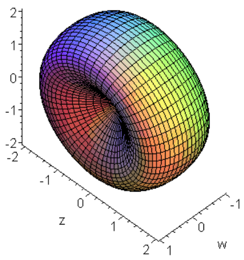


Figure 13: $(x(t), y(t)) = (\sin(2t), 2\sin(t)),\ 0 \leq t \leq 2\pi$

(a) (b)



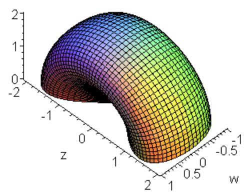


Figure 14: $(x(t), y(t)) = (\sin(2t), 2\sin(t)),\ 0 \leq t \leq \pi$

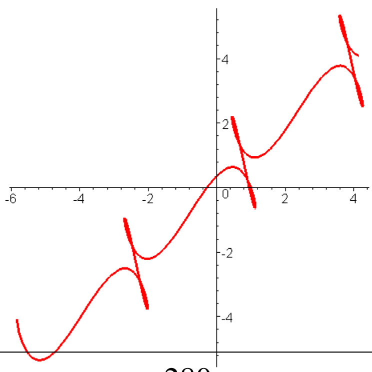


Figure 15: The Barbed Wire Curve

Figure 16: Surfaces of Revolution for the Barbed Wire Curve

Figure 17: Axes of Revolution for the Barbed Wire Curve

Figure 18: Surface of Revolution for $m = -1$

Figure 19: Surface of Revolution for $m = 0$

Figure 20: Surface of Revolution for $m = 1$

5 Summary and Final Remarks

In this paper we described a procedure for minimizing surface area when the graph of a function or a more general parametric curve is revolved about oblique lines. We did so in a manner that made use of the results in [8]. We provided several examples to illustrate the process and the interactions of the surfaces and the axes of revolution. We should point out that depending on the complexity of the function or parametric curve and the length of the interval $[a, b]$, it may not be feasible to determine the minimum surfaces of revolution due to the rather demanding computational requirements; but it is feasible to do so for many problems of interest.

Acknowledgment. The author is indebted to Wei-Chi Yang for suggesting the question of revolving graphs of functions and parametric curves about oblique lines.

6 Supplemental Electronic Materials

1. ExamplesTable.pdf, a pdf file containing a description of forty-three functions
2. Revslant.mws, a Maple worksheet to obtain the results for each of forty-three functions
3. ParametricRevslant.mws, a Maple worksheet to obtain the results for each of twenty parametric curves

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